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Time Evolution of Infinite One-Dimensional Coulomb System

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We consider one-dimensional Coulomb systems and their time evolution given by the Newton law. We give existence and uniqueness theorems for the solutions of the equations governing the systems in the thermodynamic limits.

KEY WORDS: One-dimensional Coulomb systems; nonequilibrium dynamics.

1. INTRODUCTION

This paper is based on a preprint of the same authors,[1] in which the time evolution of one-dimensional Coulomb systems was studied.

In that work we considered such systems in two different limits: the Vlasov and thermodynamic limits.

After this preprint was written, we have learnt that the Vlasov limit for those systems has been previously studied by H. Neunzert and co-workers, obtaining results similar to ours. Although these results are still unpublished we prefer not to insert the Vlasov part in this paper.

Consider a one-dimensional system of charged classical particles interacting via Coulomb forces. Let the total number of particles go to infinity, while the charge density and the total charge tend to some constant values. The limiting dynamics, if it exists, allows one to describe the time evolution of physically interesting states and hence the time-dependent thermodynamic behavior of the systems.

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This kind of infinite dynamics has already been exploited by Lanford and by Dobrushin and Fritz[2] for one-dimensional neutral particle systems, interacting via short-range forces. Other results in higher dimensions or different approaches to the problem may be found in Ref. 3.

The reason for the existence of the dynamics for short-range forces is that any kind of perturbations that are sufficiently far removed do not greatly affect the motion of a test particle. This is not the case in our context, since adding one extra particle, even very far apart, we get a significant change in the motion of the test particle, because of the long range of the forces.

Nevertheless a limit dynamics can still be defined. Following Refs. 4 and 5 we realize that the main object to study in a statistical mechanical setup for this system is the profile of the electric field instead of the distribution of the charges. What we are able to prove is that a local perturbation of the electric field does not greatly influence the evolution of the electric field in regions that are sufficiently far away. We remark that a *local* perturbation of the electric field can be realized *only* conserving the total change. This control is sufficient in order that the evolution of a very large class of states makes sense.

In the next section we solve this problem. Some probabilistic considerations that are necessary for completing our analysis are developed in the Appendix.

2. INFINITE VOLUME DYNAMICS

Consider a system of infinite point charged particles. Its evolution is described by

$$\dot{q}_i(t) = v_i(t)$$
$$m\dot{v}_i(t) = \sum_j \sigma_i\sigma_j F(q_i(t) - q_j(t)), \qquad i \in \mathbb{N} \tag{2.1}$$

with initial conditions

$$q_i(0) = q_i, \qquad v_i(0) = v_i \tag{2.2}$$

Here m denotes the mass of the particles, $\sigma_i = \pm\gamma$ their charge, and $F(q) = \mathrm{sgn}(q)$, $q \in \mathbb{R}^1$, $q \neq 0$, $F(0) = 0$.

The aim of this section is to give sense to the initial value problem (2.1) and (2.2).

In spite of the simplicity of the forces, two features of the system (2.1) make the technology developed in Ref. 2 not directly applicable. First, we deal with non-Lipschitz forces, thus some care is to be used also in defining a finite particles dynamics. Secondly the interaction has very long range and this may create problems.

We begin by discussing the second (conceptually more interesting) problem.

A quite natural approach to the above evolution problem is to find a solution of Eq. (2.1) as a limit of finite particle dynamics (after having made sense to it). Unfortunately, in general, such a limit does not exist. This is due to the fact that in adding a new particle to a finite system, a drastic perturbation arises in the motion of any other particle. This difficulty, however, can be avoided. Since we are interested, for thermodynamical reasons, in neutral systems, the natural perturbations to study are neutral too. Restricting our attention to neutral and local perturbations, we expect them to travel with finite speed.

In fact the adding of pairs of opposite new charges does not change the force acting on a distant test particle at time zero.

These considerations would allow us to control the above limiting procedure.

Let us be more precise in what we mean by the time evolution of a neutral state. We begin by giving its definition.

Let $L \in \mathbb{N}$ and ν_L be a symmetric probability measure on the space

$$\mathfrak{X}(L) = \bigcup_{n=0} \left\{ (q_i, \sigma_i, v_i)_{i=1}^n q_i \in (-L, L), v_i \in \mathbb{R}^1, \sigma = \pm\gamma \right\}$$

The neutrality condition is expressed by the fact that ν_L is concentrated on the set of all $X \in \mathfrak{X}(L)$ such that $\sum_{i=1}^n \sigma_i = 0$. In order to study the thermodynamic limit ($L \to +\infty$) it is convenient to consider the distribution of the electric field rather than the distribution of the charges. Let us introduce the space $\Omega(L) = \{E, V(E)\}$, where E, the electric field, is a step function defined in $[-L, L]$, with finitely many jumps, with range in $2\gamma\mathbb{Z}$, and such that $E(-L) = E(L) = 0$ (neutrality condition). $V(E)$ is the velocities field, $V(E) = \{v_i\}_{i=1}^n$ are the velocities of the charges located in the discontinuity points of E.

Clearly, ν_L induce a measure on $\Omega(L)$ that we denote $\tilde{\nu}_L(dE, dV)$.

As an example, the equilibrium measure at inverse temperature β and activity z is (see Refs. 4 and 5)

$$\tilde{\nu}_L(dE, dV) = \frac{P_L(dE)\,dV \left\{ \exp - \beta \left[\sum_i \frac{v_i^2}{2m} + \frac{1}{4} \int_{-L}^{L} E^2(x)\, dx \right] \right\}}{\text{norm}} \tag{2.3}$$

where $P_L(dE)$ is the Poisson measure (with parameter z) defined on the space of all the paths E [with values in $2\gamma\mathbb{Z}$, piecewise constant and satisfying $E(-L) = E(L) = 0$]. Namely, the probability of having n jumps of $\pm 2\gamma$ in $[-L, L]$ in $dq_1, \ldots, dq_n$ is given by

$$(\exp - 2zL)\frac{z^n}{n!}\, dq_1 \cdots dq_n \tag{2.4}$$

In (2.3),

$$dV = \frac{dv_1 \cdots dv_n}{n!}$$

Other nonequilibrium measures can be defined in an obvious way. For example, states with space depending activity or temperature may be defined modifying the free measure $P_L(dE)$ or the Gibbs factor in (2.3). The thermodynamic limit for the measures (2.3) (and hence, with minor modifications, for the above nonequilibrium measures) has already been studied in Refs. 4 and 5 (see also the Appendix).

The weak limit $\tilde{\nu} = \lim_{L\to\infty} \tilde{\nu}_L$, exists and is defined on the space $\Omega = \{E, V(E)\}$, where $\mathbb{R}^1 \ni x \to E(x) \in 2\gamma\mathbb{Z}$ is a step function with locally finite jumps and $V(E) = \{v_i\}_{i\in\mathbb{Z}}$ are the velocities of the discontinuities of E.

It is useful to define the electric field also in the discontinuities points as

$$E(x) = \lim_{\epsilon\to 0^+} \frac{E(x+\epsilon) + E(x-\epsilon)}{2} \tag{2.5}$$

Now we are able to state the dynamical problem in Ω. Let us come back to the particle interpretation, denoting by q_i the discontinuity points (particles) of E and σ_i the associated charge $\pm\gamma$, if the field E is jumping of $\pm 2\gamma$, respectively.

We want to find a one-parameter group of transformations $T_t : \Omega \to \Omega$, $t \in \mathbb{R}$, $\Omega \ni \omega \to T_t\omega = \omega(t)$, satisfying

$$\begin{aligned} \dot{q}_i(t) &= v_i(t) \\ m\dot{v}_i(t) &= \sigma_i E(\omega(t), q_i), \qquad i \in \mathbb{Z} \end{aligned} \tag{2.6}$$

where $E(\omega(t), q_i)$ is the electric field of $\omega(t)$ calculated at the point q_i. The above map has to be understood in the following sense. At time zero starting from $\omega = \omega(0)$ we know the position of the charges and their velocities. They evolve according to (2.6) modifying the electric field that drives the motion.

We choose with some care the initial conditions ω for which we give sense to the evolution problem (2.6). This is necessary because initial configurations ω, exhibiting rapidly increasing behavior of E and V, may create singularities in finite time (as usual in this kind of problem; see Refs. 2 and 3).

From a physical point of view, the only restriction we impose in choosing the set Ω_0 of the initial configurations is that it has to be typical for thermodynamically interesting measures.

We define

$$\Omega_0 = \{\omega \in \Omega \mid Q(\omega) < +\infty\} \tag{2.7}$$

where

$$Q(\omega) = \max(Q_1(\omega), Q_2(\omega), Q_3(\omega)) \tag{2.8}$$

and

$$Q_1(\omega) = \max\left(\sup_{x \in \mathbb{R}} \frac{|E(\omega, x)|}{\varphi(x)}, 1 \right) \tag{2.9}$$

$$Q_2(\omega) = \max\left(\sup_{i \in \mathbb{Z}} \frac{|v_i(\omega)|}{\varphi(q_i(\omega))}, 1 \right) \tag{2.10}$$

$$Q_3(\omega) = \max\left(\sup_{x} \sup_{a > \varphi(x)} \frac{N(\omega \mid I(x,a))}{2a}, 1 \right) \tag{2.11}$$

where $\varphi(x) = \max(1, \log|x|)$, $I(x,a) = (x-a, x+a)$, and $N(\omega \mid A)$ is the number of discontinuities (particles) in the set $A \subset \mathbb{R}$.

It is proved in the Appendix, that the set Ω_0 is of full measure with respect to the equilibrium measure $\tilde{\nu}$ (and also with respect to other nonequilibrium measures as those discussed above).

For all $\omega \in \Omega$ we define the map $\omega \to \omega^L(t) \in \Omega$ as the solution of the following initial value problem:

$$\begin{aligned} q_i(t) &= q_i(0) + \int_0^t v_i(s)\, ds \\ v_i(t) &= v_i(0) + \int_0^t ds \left\{ \sum_j \sigma_i \sigma_j F(q_i(s) - q_j(s)) \right. \\ &\qquad \left. \times q_j(s) \in (-L, L) + \sigma_i \frac{E_r(\omega) + E_l(\omega)}{2} \right\} \end{aligned} \tag{2.12a}$$

if i such that $q_i(0) \in (-L, L)$; and

$$\begin{aligned} q_i(t) &= q_i(0) \\ v_i(t) &= v_i(0) \end{aligned} \tag{2.12b}$$

otherwise. Here $E_l(\omega) = E(\omega, -L)$, $E_r(\omega) = E(\omega, +L)$.

The above problem corresponds to the Hamiltonian time evolution of the particles inside $[-L, L]$ under the action of their own field and the field of the external ones thought of as fixed. We have put the evolution problem in integral form, because the discontinuity of the force makes the differential version of the problem meaningless. We assume also elastic boundary conditions in $\pm L$.

It turns out that the only source of nonuniqueness for the solutions of problem (2.12) lies in the possibility that a pair of particles collides with zero relative velocity. In fact this initial condition may give rise to two different time evolutions. We solve this ambiguity by "choosing" the solution in which the particles will separate after the collision. This corresponds to defining the solution of (2.12) as the limiting solution, for $\epsilon \to 0$, of a regularized dynamics given by a force mollified in an ϵ neighborhood of the origin.

In any case this pathology is not relevant from a physical point of view, since the set of initial data exhibiting such a behavior has Lebesgue measure zero.

The solution of the problem (2.12) is explicitly obtained in the space-time $[-L, L] \times \mathbb{R}$ as a collection of continuous curves each of them being a sequence of arcs of parabolas. Thus for any $\omega \in \Omega_0$ we shall look for a limit point of $\omega^L(t)$ as $L \to \infty$. It exists in view of the estimate of the next theorem. Denoting $\omega = \{q_i, \sigma_i, v_i\}$, $\omega^L(t) = \{q_i^L(t), \sigma_i, v_i^L(t)\}$, we have the following:

Theorem 2.1. Let $\omega \in \Omega_0$. There exists a positive, continuous, increasing function $h(\omega, \cdot)$, such that

$$|q_i^L(t) - q_i(0)| \leqslant h(\omega, |t|)\varphi(q_i(0)), \qquad t \in \mathbb{R} \tag{2.13}$$

Proof. Fixed $\omega \equiv \{q_i, \sigma_i, v_i\} \in \Omega_0$ and denoting $\omega^L(t) \equiv \{q_i(t), \sigma_i, v_i(t)\}$, (dropping the index L for notational simplicity) we have

$$E(\omega^L(s), q_i(s)) = E(\omega, q_i(s)) + 2\Phi(\omega^L(s), q_i(s)) \tag{2.14}$$

where $\Phi(\omega^L(s), x)$ is the total charge crossing the point x from right to left, minus the total charge crossing the same point from left to right, during the time s, following the dynamics $\omega^L(s)$.

We define for $t \geqslant 0$:

$$\begin{aligned} s_i(t) &= \sup_{0 \leqslant s \leqslant t} |q_i(t) - q_i(0)| \\ d(t) &= \sup_i \frac{s_i(t)}{\varphi(q_i)} \end{aligned} \tag{2.15}$$

and denote by $\tilde{q}_i(t)$ the solution of the equation

$$\tilde{q}_i(t) = d(t)\varphi(\tilde{q}_i(t)) + q_i(t) \tag{2.16}$$

Then the particles that can reach up to the time t, the geometrical point $q_i(t)$, lie, at time zero, at most in the interval $[2q_i(t) - \tilde{q}_i(t), \tilde{q}_i(t)]$.

Hence

$$\left|\Phi\left(\omega^L(s), q_i(s)\right)\right| \leqslant \gamma N\left(\omega \,|\, \left[q_i(s) - d(s)\varphi(\tilde{q}_i(s)), q_i(s) + d(s)\varphi(\tilde{q}_i(s)) \right]\right) \tag{2.17}$$

In virtue of (3.14) and (3.15)

$$\begin{aligned} s_i(t) &\leqslant |p_i(0)|t + \gamma\int_0^t (t-s)\left|E(\omega(s), q_i(s))\right| ds \\ &\leqslant |p_i(0)|t + \gamma\int_0^t (t-s)\left|E(\omega, q_i(s))\right| ds \\ &\quad + 2\gamma\int_0^t (t-s)\left|\Phi\left(\omega^L(s), q_i(s)\right)\right| ds \\ &\leqslant |p_i(0)|t + \gamma\int_0^t (t-s) Q_1(\omega)\varphi(q_i(s))\, ds \\ &\quad + 2\gamma^2\int_0^t (t-s) Q_3(\omega)\varphi(q_i(s))\, ds \\ &\quad + 2\gamma^2\int_0^t (t-s) Q_3(\omega) d(s)\varphi(\tilde{q}_i(s))\, ds \end{aligned} \tag{2.18}$$

where we have used (2.14), (2.17), (2.9), (2.11).

Hence

$$\begin{aligned} d(t) \leqslant\ & Q_2(\omega)t + \left[\gamma Q_1(\omega) + 2\gamma^2 Q_3(\omega)\right] \\ & \times \int_0^t (t-s)\left[\sup_i \frac{\varphi(q_i(s))}{\varphi(q_i(0))}\right] ds \\ & + 2\gamma^2 Q_3(\omega)\int_0^t (t-s)\left[\sup_i \frac{\varphi(\tilde{q}_i(s))}{\varphi(q_i(0))}\right] d(s)\, ds \end{aligned} \tag{2.19}$$

A bound for $\tilde{q}_i(t)$ may easily be obtained using the inequality $\varphi(x) \leqslant 2(x+1)^{1/2}$ in (2.16) and then bounding the solution

$$|\tilde{q}_i(t)| \leqslant 8\left[d(t)^2 + |q_i(t)| + 1\right] \tag{2.20}$$

Using again (2.16)

$$\begin{aligned} |\tilde{q}_i(t)| &\leqslant d(t)\left[2\varphi(d(t)) + \varphi(q_i(t)) + k_1\right] + |q_i(t)| \\ &\leqslant k_2(t)\left[\varphi(d(t)) + \varphi(q_i(t))\right] + |q_i(t)| \end{aligned} \tag{2.21}$$

where $k_1 = \varphi(8) + \varphi(1)$ and $k_2 = 2k_1$ [having used $\varphi(x+y) \leqslant \varphi(x) + \varphi(y)$].

Furthermore

$$\begin{aligned}\varphi(\tilde{q}_i(t)) &\leqslant \varphi(k_2) + \varphi(d(t)) + \varphi(q_i(t)) + \varphi(\varphi(d(t))) \\ &\quad + \varphi(\varphi(q_i(t))) \\ &\leqslant k_3\big[\varphi(d(t)) + \varphi(q_i(t))\big] \end{aligned} \tag{2.22}$$

where $k_3 = \varphi(k_2) + 4$ [having used $\varphi(x) \leqslant |x| + 1$].

Finally

$$\frac{\varphi(q_i(t))}{\varphi(q_i(0))} \leqslant 1 + \frac{\varphi(d(t))\varphi(q_i(0))}{\varphi(q_i(0))} \leqslant 2 + \varphi(d(t)) \tag{2.23}$$

and hence

$$\frac{\varphi(\tilde{q}_i(t))}{\varphi(q_i(0))} \leqslant k_4\varphi(d(t)) \tag{2.24}$$

where $k_4 = 4k_3$.

Thus, by (2.19), putting $k_5 = 2(\gamma + 2\gamma^2)$, $k_6 = 2\gamma^2 k_4$

$$\begin{aligned} d(t) &\leqslant Q(\omega)t + k_5 Q(\omega)\int_0^t (t-s)\varphi(d(s)) \\ &\quad + k_6 Q(\omega)\int_0^t (t-s)d(s)\varphi(d(s)) \\ &\leqslant Q(\omega)t + (k_5 + k_6)Q(\omega)\int_0^t (t-s)\big[d(s)+1\big]\varphi(d(s)) \end{aligned} \tag{2.25}$$

and the thesis follows by observing that there exists a global solution for the differential problem associated to the integral inequality (2.25). ■

The control (uniform in L) on the displacements, allow us to bound the growth of the electric field in any point [see (3.14) and (3.17)] and hence the force acting (up to some fixed time t) on each particle.

Theorem 2.2. There exists a one-parameter group of transformations

$$\Omega_0 \ni \omega \to \omega(t) \in \Omega_0, \quad \omega = \omega(0) = \{q_i, \sigma_i, v_i\}_{i\in\mathbb{Z}}, \quad \omega(t) = \{q_i(t), \sigma_i, v_i(t)\}_{i\in\mathbb{Z}}$$

satisfying the integral equations

$$q_i(t) = q_i + v_i t + \int_0^t ds(t-s)\sigma_i E(\omega(s), q_i(s)), \qquad i \in \mathbb{Z} \tag{2.26}$$

Moreover, there exists a positive, increasing, continuous function of $|t|$,

$h(\omega, \cdot)$ and a sequence of integers $\{L_k\}_{k=1}^{\infty}$, such that

$$|q_i(t) - q_i(0)| \leqslant h(\omega, |t|)\log q_i \tag{2.27}$$

$$\lim_{k\to\infty} q_i^{L_k}(t) = q_i(t)$$
$$\lim_{k\to\infty} v_i^{L_k}(t) = v_i(t), \qquad i \in \mathbb{Z}, \quad t \in \mathbb{R} \tag{2.28}$$

uniformly in t on compact sets.

The limiting solution $\omega(t) = \{q_i(t), \sigma_i, v_i(t)\}$ is the unique solution satisfying (2.27).

Proof.

Existence. As a consequence of the bound (2.13) we get a uniform (in L) control on the forces which allows us to apply the Ascoli–Arzelà theorem to obtain (2.28). Obviously $\omega(t) = \{q_i(t), \sigma_i, v_i(t)\}$ satisfy (2.26) and (2.27). Moreover $\omega(t) \in \Omega_0$ as a consequence of (2.27).

Uniqueness. Let $\omega \to \omega^s(t)$, $s = 1,2$ be two solutions of (2.26) satisfying (2.27) and look at the ith particle. Then $q_i^1(t)$ and $q_i^2(t)$ coincide up to some instant t_1 in which the ith particle collides with some other particle (say j_1) with the following properties:

$$q_{j_1}^1(t_1) = q_i^s(t_1), \qquad q_{j_1}^2(t_1) \neq q_i^s(t_1), \qquad s = 1,2 \tag{2.29}$$

This is because the only possibility for the ith particle to bifurcate its motion is to feel different external field and this means that it suffers different collisions in the two dynamics.

Let us follow backward the motion of the particle j_1. The particle j_1 has already collided, at some first time t_2, with a particle $j_2 \neq i$ which has already bifurcated its motion and so on.

Let $\{t_i\}_{i=1,2\ldots}$ be such collision times. They cannot be a finite number. If so, there would be an instant t_n and a particle j_n which has bifurcated in $(0, t_n)$ without colliding in such time with any other bifurcated particle. But this is impossible. Since the collision times are an infinite number, let us denote $\bar{q}(t) = q_{j_k}(t)$, $t_{k+1} < t \leqslant t_k$. Then

$$\bar{q}(s) = q_i(t_1) - \left(\sum_{k=1}^{h} \int_{t_{k+1}}^{t_k} d\tau\, v_{j_k}(\tau) + \int_s^{t_{h+1}} d\tau\, v_{j_{h+1}}(\tau) \right) \tag{2.30}$$

for $s \in (t_{h+2}, t_{h+1}]$. Since $|v_{j_k}(\tau)| \leqslant Q_2(\omega(\tau))\varphi(q_{j_k}(\tau))$, by putting

$$u = \sup_{0<s\leqslant t_1} |\bar{q}(s) - q_i(t_1)|, \qquad Q = \sup_{0<s\leqslant t_1} Q(\omega(s))$$

we have by (2.30)

$$u \leqslant |t_1|\, Q\, \{\varphi(u) + \varphi(q_i(t_1))\} \tag{2.31}$$

This implies $u < +\infty$. Since $q_{j_k}(0) \neq q_{j_h}(0)$ for all $h \neq k$, in the strip $[q_i(t_1) - u, q_i(t_1) + u] \times [0, t_1]$ will fall infinitely many particles. This is clearly impossible, since $\omega(t) \in \Omega_0$ and the proof of Theorem 2.2 is achieved. ■

Remark. In virtue of the uniqueness theorem one can get the convergence for any sequence in (2.28).

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APPENDIX

We shall prove that

$$\tilde{\nu}(\Omega_0) = 1 \tag{A.1}$$

where Ω_0 is defined in (2.7) and $\tilde{\nu}$ is the weak local limit of $\tilde{\nu}_L$ defined in (2.3).

Let us recall some facts about such a limit (see Ref. 5 for details). Ignoring the momenta that are independently distributed, one defines

$$\tilde{\mu}_L(f) = \frac{\int P_L(dE)\left\{\exp\left[-\frac{\beta}{4}\int_{-L}^{L} E^2(x)\,dx\right]\right\} f(E)}{\int P_L(dE)\left\{\exp\left[-\frac{\beta}{4}\int_{-L}^{L} E^2(x)\,dx\right]\right\}} \tag{A.2}$$

We introduce the measure $P_{[a,b]}(dE \mid u, v), u, v \in 2\gamma\mathbb{Z}$, that is the Poisson measure on the jump process starting at a, with value u, arriving in b with value v. [See (2.4).] By definition we have $P_{[-L,L]}(dE \mid 0,0) = P_L(dE)$. $P_{[a,b]}(dE \mid u, v)$ is normalized to

$$\sum_{n \geqslant |u-v|} z^n \frac{|b-a|^n}{n!} \exp - z|b-a|$$

A semigroup on $l_2(2\gamma\mathbb{Z})$ may be defined via the following kernel:

$$(\exp - \mathfrak{L}x)(u, v) = \int P_{[0,x]}(dE \mid u, v) \exp\left[-\frac{\beta}{4}\int_0^x E(y)^2\,dy\right] \tag{A.3}$$

It follows by the Feynman–Kac formula

$$(\mathfrak{L}\psi)(u) = \left[\left(-z\Delta + \frac{\beta}{4}u^2\right)\psi\right](u), \qquad \psi \in l_2(2\gamma\mathbb{Z}) \tag{A.4}$$

where

$$(\Delta\psi)(u) = \psi(u + 2\gamma) + \psi(u - 2\gamma) - 2\psi(u) \tag{A.5}$$

Defining

$$\chi_v(u) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{otherwise} \end{cases} \tag{A.6}$$

and denoting by $(\cdot, \cdot)$ the inner product on $l_2(2\gamma\mathbb{Z})$, one defines

$$\begin{aligned} \tilde{\mu}(F) &= \lim_{L\to\infty} \tilde{\mu}_L(F) \\ &= \lim_{L\to\infty} \frac{\left(\chi_0, e^{-\mathfrak{L}(x+L)} f e^{-\mathfrak{L}(L-x)}\chi_0\right)}{\left(\chi_0, e^{-2\mathfrak{L}L}\chi_0\right)} \end{aligned} \tag{A.7}$$

for all functions $F(E) = f(E(x))$, $f \in l_2(2\gamma\mathbb{Z})$.

Such limit exists by the Krein–Rutman theorem and moreover:

$$\tilde{\mu}(F) = (\psi, f\psi) \tag{A.8}$$

where ψ denote the nondegenerate ground state of the positive operator $\mathfrak{L}$.

We claim

$$\tilde{\mu}(\exp \delta|E(x)|) \leqslant C < +\infty \tag{A.9}$$

for sufficiently small $\delta > 0$.

Proof. The eigenvalue equation is

$$\left[(-z\Delta + \alpha)\psi\right](u) + \tfrac{1}{4}\beta u^2\psi(u) = \left[(\lambda + \alpha)\psi\right](u) \tag{A.10}$$

for $\alpha \geqslant 0$. $\lambda > 0$ is the eigenvalue associated to ψ.

Hence

$$\psi(u) = \int_0^\infty dt\left[\exp(-\alpha + z\Delta)t\right]\left[(\lambda + \alpha) - \frac{\beta}{4}u^2\right]\psi(u) \tag{A.11}$$

Expanding in power series the off diagonal part of Δ

$$\psi(u) = \int_0^\infty dt \exp(-\alpha - 2z)t \sum_{v\in 2\gamma\mathbb{Z}} \sum_{\eta: u\to v} \times \frac{(zt)^{|\eta|}}{|\eta|!}\left[(\lambda + \alpha) - \frac{\beta}{4}v^2\right]\psi(v) \tag{A.12}$$

where $\eta : u \to v$ is a path in $2\gamma\mathbb{Z}$ starting from u and arriving in v with $|\eta|$ jumps of length 2γ. Since $\psi(u) \geqslant 0$:

$$\begin{aligned} \psi(u) &\leqslant \sum_{v\leqslant[4(\lambda+\alpha)/\beta]^{1/2}} \sum_{n\geqslant|u-v|} \frac{(2z)^n}{n!}\int_0^\infty dt \exp\left[-(\alpha + 2z)t\right]t^n\cdot(\lambda + \alpha)\|\psi\|_\infty \\ &\leqslant \sum_{v\leqslant[4(\lambda+\alpha)/\beta]^{1/2}} \left[2/(\alpha + 2z)\right] \sum_{n\geqslant|u-v|} \left(\frac{4zk}{\alpha + 2z}\right)^n \end{aligned} \tag{A.13}$$

for some $k > 0$. By choosing α large enough, it follows that $\psi(u)$ has, at least, exponentially decay and hence the claim.

We define

$$B = \left\{ \omega \in \Omega \mid \sup_{x \in \mathbb{Q}} \frac{|E(\omega, x)|}{\varphi(x)} < +\infty \right\} \tag{A.14}$$

where $\mathbb{Q}$ is the set of rational numbers. We shall prove $\tilde{\mu}(B) = 1$. This implies $\tilde{\nu}(\{\omega \mid Q_1(\omega) < +\infty\}) = 1$.

It is enough to prove that

$$\tilde{\mu}(B_\epsilon) = 1 \tag{A.15}$$

where

$$B_\epsilon = \left\{ \omega \in \Omega \mid \sup_{x \in \epsilon\mathbb{Z}} \frac{|E(\omega, x)|}{\varphi(x)} < +\infty \right\} \tag{A.16}$$

We have

$$\begin{aligned} B_\epsilon &= \bigcup_{m \in \mathbb{N}} \{\omega \mid |E(\omega, x)| \leqslant m\varphi(x), x \in \epsilon\mathbb{Z}\} \\ B_\epsilon^c &= \bigcap_{m \in \mathbb{N}}{}' \{\omega \mid |E(\omega, x)| > m\varphi(x) \text{ for some } x \in \epsilon\mathbb{Z}\} \\ &= \bigcap_{m \in \mathbb{N}} \bigcup_{x \in \epsilon\mathbb{Z}} \{\omega \mid |E(\omega, x)| > m\varphi(x)\} \end{aligned} \tag{A.17}$$

Hence, by the Tchebycev inequality,

$$\begin{aligned} \tilde{\mu}(B_\epsilon^c) &\leqslant \lim_{m \to \infty} \sum_{x \in \epsilon\mathbb{Z}} \mu(\{\omega \mid |E(\omega, x)| > m\varphi(x)\}) \\ &\leqslant \lim_{m \to \infty} \sum_{x \in \epsilon\mathbb{Z}} C \exp[-\delta m \varphi(x)] \\ &\leqslant \lim_{m \to \infty} \left[C \frac{2\exp[-\delta m + 1]}{\epsilon} + \sum_{\substack{x \in \epsilon\mathbb{Z} \\ |x| > e}} |x|^{-\delta m} \right] = 0 \end{aligned} \tag{A.18}$$

Thus (A.15) is proved.

To complete the proof of (A.1) we need the following estimates:

$$\int \tilde{\nu}(d\omega) \exp \delta_1 T(\omega \mid [a, b]) \leqslant \bar{c}_1 \exp c_1 |b - a| \tag{A.19}$$

$$\int \tilde{\mu}(dE) \exp \delta N(E \mid [a, b]) \leqslant \bar{c}_2 \exp c_2 |b - a| \tag{A.20}$$

for sufficiently small $\delta_1 > 0$, and $\delta > 0$ and $c_1, c_2, \bar{c}_1, \bar{c}_2$ large enough. Here

$$T(\omega\,|\,[a,b]) = \sum_{i\,:\,q_i\in[a,b]} \frac{v_i^2}{2m}$$

and $N(E\,|\,[a,b])$ is the number of jumps of E in $[a,b]$.

In fact, using union-intersection tricks as above, by (A.19) and (A.20) one deduces

$$\tilde{\nu}\left(\left\{\omega\,|\sup_x \sup_{a>\varphi(x)} \frac{N(\omega\,|\,I(x,a))}{2a} < +\infty\right\}\right) = 1 \qquad \text{(A.21)}$$

$$\tilde{\nu}\left(\left\{\omega\,|\sup_x \sup_{a>\varphi(x)} \frac{T(\omega\,|\,I(x,a))}{2} < +\infty\right\}\right) = 1 \qquad \text{(A.22)}$$

and by inequality

$$\frac{|v_i|}{2m} \leqslant \left[\sup_x \sup_{a>\varphi(x)} \frac{T(\omega\,|\,I(x,a))}{2}\varphi(q_i)\right]^{1/2} \qquad \text{(A.23)}$$

it follows

$$\tilde{\nu}(\{\omega\,|\,Q_2(\omega) < +\infty\}) = \tilde{\nu}(\{\omega\,|\,Q_3(\omega) < +\infty\}) = 1 \qquad \text{(A.24)}$$

Thus it remains to prove (A.19) and (A.20).

Integrating explicitly on the momenta,

$$\int \tilde{\nu}(d\omega)\exp\{\delta_1 T(\omega\,|\,[a,b])\} \leqslant \int \tilde{\mu}(dE)\exp\delta N(E\,|\,[a,b]) \qquad \text{(A.25)}$$

for some $\delta > 0$, thus (A.19) follows from (A.20).

Proof of (A.20). If $[-L,L]\supset[a,b]$:

$$\begin{aligned}(\chi_0, e^{-2\mathcal{E}L}\chi_0)\mu_L(e^{\delta N(\cdot\,|\,[a,b])}) &= \sum_{u,v\in 2\gamma\mathbb{Z}} \int P_{[-L,a]}(dE\,|\,0,u)\exp\left[-\frac{\beta}{4}\int_{-L}^{a} E^2(x)\,dx\right]\\ &\times \int P_{[a,b]}(dE\,|\,u,v)\exp\left\{-\frac{\beta}{4}\int_a^b E^2(x)\,dx + \delta N(E\,|\,[a,b])\right\}\\ &\times \int P_{[b,L]}(dE\,|\,v,0)\exp\left\{-\frac{\beta}{4}\int_b^L E^2(x)\,dx\right\} \qquad \text{(A.26)}\end{aligned}$$

The second integral on the right-hand side of (A.26) is bounded by

$$\sum_{n\geqslant|u-v|} \frac{e^{\delta n}|b-a|^n\exp(-z|b-a|)}{n!} \leqslant \exp\left[z(e^\delta - 1)|b-a|\right] \qquad \text{(A.27)}$$

Hence

$$\mu(e^{\delta N(\cdot\,|[a,b])}) = \lim_{L\to\infty} \sum_{u,v\in 2\gamma\mathbb{Z}} \exp\left[z(e^{\delta}-1)|b-a|\right] \times \frac{\left(\chi_0, e^{-\mathfrak{L}(a+L)}\chi_v\right)\left(\chi_u, e^{-\mathfrak{L}(L-b)}\chi_0\right)}{\left(\chi_0, e^{-2\mathfrak{L}L}\chi_0\right)}$$

$$= \sum_{u,v\in 2\gamma\mathbb{Z}} \psi(u)\psi(v)\exp\left[\lambda + z(e^{\delta}-1)\right]|b-a| \quad \text{(A.28)}$$

This proves (A.20).

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